SOLUTION METHOD FOR TEHD FLUID CONTACT PROBLEMS BASED ON P-FEM MODEL

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SUMMARY
For solving contact problems under fluid film lubrication condition when the properties of the lubricant depends on the temperature and the pressure several methods have been already developed, e.g. the well-known fast approach scheme published by Houpert & Hamrock (1986) [9]. However, in case of these solving systems same disadvantages can be found such as the very high point number caused by the finite difference method background. Unfortunately, only few solutions were published based on FEM despite it permits the using of polynomial approximation for powerful calculation of the generalized Reynolds equation without smooth mesh. So the development of a p-version FEM model for calculating the film shape, the pressure and temperature distribution in case of TEHD (Thermo-Elasto Hydrodynamic) problems seems to be timely.

Keywords: TEHD lubrication, p-Finite-Element Method, Penalty-Cavitation

1 INTRODUCTION
As well known, after Osborne Reynolds (1886) [1], who approached the lubricating phenomenon analytically, presented the theory of hydrodynamic lubrication and the governing partial differential equation for the pressure field, many complex problems in fluid-film lubrication have been studied. Rolling-sliding machines such as gears, cams and followers, and bearings are often subjected to high loads, high speeds and high slip conditions. The elastohydrodynamic lubrication of rolling-sliding contacts has been investigated extensively by Dowson (1961) [3], Sternlicht et al. (1961) [4], Cheng (1970) [6], Houpert and Hamrock (1986) [9], Lubrecht et al. (1986) [10], and Sadeghi (1990) [12]. Wolff et al. (1992) [14] examined the influence of the different viscosity models on the results. He et al. advised to use the modified WLF formula for TEHD problems, which has the best fitting of experimentally measured viscosity. So it will be used for the calculation, although solving a real problem is not the aim of this work now.

These works based on the solution of the generalized Reynolds equation by finite difference method. However, in many practical problems involving complex film thickness, irregular bearing shapes and boundary conditions, the finite difference methods are clumsy and difficult to use. Lubrication analysis began looking into the finite element formulation in the sixties by Reddi (1969) [5], the seventies by Rohde and Oh (1975) [8], and they found the method promising and reliable. Freund and Tieu (1993) [15] used h-FEM for solving the generalised Reynolds equation. In these investigations, the traditional h-version finite element method (h-FEM) had been used in the formulation of lubrication problems till 1991 when Nguyen [13] reported an analysis with using p-version finite element method but the geometry and the oil properties were assumed to be constant. However the p-approximation concept was introduced by Babuska and Szabó. (1979) [16], but most p-version formulation appear in the literature deal with problems in solid mechanics and heat transfers. In this paper the application possibility of p-FEM for TEHD problem is presented. In this paper the application possibility of p-FEM for TEHD problem is presented.

2 HYDRODYNAMIC PROBLEM
For calculating the contact pressure due to fluid film lubrication the generalized Reynolds equation was developed by Dowson (1961) [3] as a partial differential equation take account the changes the viscosity across the film thickness. Based on the concepts of it, the weak integral form of the generalized Reynolds equation is:

\[
\int_{A} \nabla_{xy} w \cdot \Phi \cdot \nabla_{xy} p \cdot dA - \int_{A} \nabla_{xy} w \cdot \Psi \cdot dA - \int_{A} w \cdot \Omega \cdot dA + \int_{s} w \cdot q_{s} \cdot ds = 0
\]

where:

\[
\Phi = F_{2} + G_{1}
\]

\[
\Psi = h \cdot \rho \bigg|_{z=h} \cdot u \bigg|_{z=h} - \frac{F_{3} - G_{2}}{F_{0}} \bigg( u \bigg|_{z=h} - u \bigg|_{z=0} \bigg)
\]

\[
\Omega = p \bigg|_{z=h} \cdot v \bigg|_{z=h} \cdot \nabla_{x,y} \cdot h + p \bigg|_{z=0} \cdot v \bigg|_{z=0}
\]

\[
u^{T} = [u_{x}, u_{y}]^{T}
\]

\[
F_{i}G_{i} : \text{viscosity-density functions according to Dowson (1961) [3]}
\]

\[
F_{0} = \int_{0}^{h} \frac{d(z)}{\eta} \eta
\]

\[
F_{1} = \int_{0}^{h} \frac{z \cdot dz}{\eta}
\]

\[
F_{2} = \int_{0}^{h} \frac{p \cdot z^{2} \cdot dz}{\eta}
\]

\[
F_{3} = \int_{0}^{h} \frac{p \cdot z^{2} \cdot dz}{\eta}
\]
\[
G_1 = \int_{z=0}^{h} \frac{\partial p}{\partial z} \frac{1}{\eta} \frac{\partial u}{\partial z} - G_2 \cdot \frac{F_1 \cdot G_3}{F_0} \tag{9}
\]
\[
G_2 = \int_{z=0}^{h} \frac{\partial p}{\partial z} \frac{1}{\eta} \frac{\partial u}{\partial z} \tag{10}
\]
\[
G_3 = \int_{z=0}^{h} \frac{\partial p}{\partial z} \frac{1}{\eta} \frac{\partial u}{\partial z} \tag{11}
\]
\[w: \text{a weight function}, h: \text{gap size in the } z \text{ direction}, A: \text{contact surface region}, s: \text{boundary of } A, q: \text{lubricant in or outlet. In case of rolling-sliding contacts without dynamic load } \Omega = 0 \text{ and } G_i \text{ are negligible.}
\]

### 3 THERMODYNAMIC PROBLEM

The thermodynamic problem of the fluid-film contact is solvable by FEM when the temperature is variable across the fluid film. From the energy equation, the weak integral model is also developable:

\[
\int_V \left( \frac{\lambda}{\rho \cdot c_v} \nabla \cdot \nabla T \right) dV + \int_V \nabla T \cdot \nabla T dV - \int_A q_n \cdot dA - \int_V \frac{D}{\rho \cdot c_v} \cdot dV = 0
\tag{12}
\]

where

\[
D = \eta \left( \frac{\partial u_x}{\partial z} \right)^2 + \left( \frac{\partial u_y}{\partial z} \right)^2
\tag{13}
\]

dissipation function according to Dowson (1961) [1],

\[
q_n = \left( \frac{\lambda}{\rho \cdot c_v} \nabla T \right) \cdot n
\tag{14}
\]

the heat transfer through the surface,

\[w: \text{a weight function}, V: \text{volume region between the contact surfaces}, A: \text{surface of } V.
\]

The oil exit boundaries all have a free temperature condition. For calculating the temperature of the contact surfaces the Carslaw & Jaeger (1959) [2] model was used as the substructure model when the analytical expression can be joined to the FEM solution by least squares approximation. According to this theory the boundary conditions are:

- at lubricant inlet and outlet boundaries:

\[
q_n = 0
\tag{15}
\]

at the “s” contact body surface, in case of line contact, the following function of temperature:

\[
\vartheta_s(x) = \frac{u_s(x)}{2 \kappa_s} \cdot \frac{1}{\pi \cdot \rho_s \cdot c_s \cdot \kappa_s} \cdot e^{-\frac{2 \kappa_s}{u_s(x-x_s)}},
\tag{16}
\]

\[
\cdot K_0 \left[ \frac{u_s(x-x_s)}{2 \cdot \kappa_s} \right] \cdot dx + \vartheta_0
\]

### 4 FILM SHAPE

For simple model, the film shape can be calculated as a superposition of the original geometry, the displacement of a rigid surface and the deformation of a half-space under pressure.

The classical approach to finding the stresses and displacement in an elastic half-space due to surface traction was presented by Boussinesq (1885) and Cerruti (1882) and was developed by Love (1952). For this case, Johnson (1985) [11] reported a simple solution. Based on this, the deformation of the surface of the half-space under the action of a normal pressure:

\[
\delta(x', y') = \frac{2}{4 \pi E} \cdot \int_A \frac{1}{\sqrt{(x-x')^2 + (y-y')^2}} \cdot dA
\tag{17}
\]

or in case of line contact:

\[
\delta(x') = \frac{2}{\pi E_s} \cdot \int_s \frac{p \cdot \ln(x' - x)^2}{2} \cdot dx
\tag{18}
\]

After deformation, the film shape:

\[h = h_0 + \Delta_{rigid} + \delta
\tag{19}
\]

where \(\Delta_{rigid}\) is the displacement as a rigid surface, \(h_0\) is the undeformed gap size, \(E\)’ is the reduced Young modulus, \(A\) is the contact surface region and \(s\) is the contact line (in case of line contact).

### 5 VISCOSITY

As it was mentioned in the introduction, the modified WFL viscosity formula is used in this paper according to Wolff et al. (1992) [14].

Modified WLF equation:

\[
\log(\eta_{WLF}) = \log(\eta_0) - \frac{C_1}{1 + \frac{C_2}{1}}
\tag{20}
\]

where:

\[\eta_0: \text{viscosity at reference temperature}
\]

\[T_0: \text{reference temperature at atmospheric pressure}
\]

\[A_1: \text{constants for the reference temperature}
\]

\[B_1: \text{constants for the thermal expansion coefficient in liquid state}
\]

\[C_1: \text{WLF constants at atmospheric pressure}
\]

### 6 CAVITATION

The polynomial approximation makes the opportunity of replacement of the smooth mesh with a rough one. On the other hand applying rough mesh the end of the contact can locates far from the boundaries of the elements. This fact makes to be important to manage the cavitation inside elements despite deactivating the elements where the cavitation occurs. For this purpose, it is possible to modify the viscosity and the heat transfer coefficient of the lubricant in the following way in other to satisfy the cavitation boundary condition:

\[
\eta = \gamma(p) \cdot \eta_{WLF}
\tag{21}
\]

\[
\lambda = \gamma(p) \cdot \lambda_0
\tag{22}
\]

\[
\gamma(p) = \begin{cases} p > p_0, \gamma = 1 \\ p \leq p_0, \gamma = c \end{cases}
\tag{23}
\]

where \(c\) is the penalty parameter (eg.:10^{-2} \ldots 10^{-4})

Since \(\gamma(p) = \gamma(p(x,y))\) function is assumed to be constant in the z direction the viscosity-density functions should be modified as the following:

\[
F_{0m} = \frac{1}{\gamma} \int_0^h \frac{dz}{\eta_{WLF}}
\tag{24}
\]
Therefore:

\[ \Phi = \frac{1}{\gamma} (F_2 + G_1) \]  

(25)

\[ D = \gamma \cdot \eta \cdot \text{WLF} \left( \left( \frac{\partial u_x}{\partial x} \right)^2 + \left( \frac{\partial u_y}{\partial z} \right)^2 \right) \]  

(26)

The modification of the other functions is not practical.

7 LOADCASE

The integral of the pressure over the contact area should be equal with the load.

\[ W = \int \rho \cdot dA \]  

(27)

or

\[ W = \int \rho \cdot dx \]  

(28)

W is the line load or the line load. Loadcase can be satisfied if the \( \Delta_{\text{eqd}} \) is a variable.

8 FINITE ELEMENT FORMULATION

Since line contact was analysed in this paper, the parameters in the y direction were assumed to be constant; only one dimensional approximation is needed for the pressure, one dimensional interpolation for the film shape and two dimensional approximation was used for the thermal problems.

In case of hydrodynamic problem the p-FEM formulation:

\[ p(\xi) = \frac{2}{1} N(\xi)_j \cdot P^e_i + \sum_{j=1}^{p^e_{\text{d}}} \tilde{N}(\xi)_j \cdot \tilde{a}^e_{pj} = \]  

(29)

\[ h(\xi) = \frac{2}{1} N(\xi)_j \cdot H^e_i + \sum_{j=1}^{p^e_{\text{d}}} \tilde{N}(\xi)_j \cdot \tilde{a}^e_{kJ} = \]  

(30)

\[ x(\xi)^e = \frac{2}{1} N(\xi)_j \cdot X^e_{pi} = N(\xi)_j \cdot X^e_p = \]  

(31)

\[ \tilde{N}^e_p = \begin{bmatrix} N \end{bmatrix}, \tilde{N}_p^e = \begin{bmatrix} N \end{bmatrix}, \tilde{P}^e = \begin{bmatrix} P \end{bmatrix} \]  

(32)

\[ \tilde{N}_h^e = \begin{bmatrix} N \end{bmatrix}, \tilde{H}_h^e = \begin{bmatrix} H \end{bmatrix} \]  

(33)

\[ \tilde{P}^e = \begin{bmatrix} P \end{bmatrix}, \tilde{a}^e = \begin{bmatrix} a \end{bmatrix} \]  

(34)

\[ \tilde{H}^e = \begin{bmatrix} H \end{bmatrix}, \tilde{a}_h^e = \begin{bmatrix} a \end{bmatrix} \]  

(35)

where \( p^e \) is the order of the approximation polynomial of the pressure, \( p^e \) is the order of the interpolation polynomial of the film shape on an element; \( P^e_i \) are the pressure values, \( H^e_i \) are the film sizes at the ends of an element; \( a^e \) are the pressure approximation constants, \( a^e_h \) are the film shape interpolation constants between the ends on an element; \( N_i \) are the h-approximation functions, \( N^p \) are the Legendre functions of the polynomial approximation according to Páczelt (1994) [16]; \( X^e_p \) are the x coordinates of the end of an element. Since the Rayleigh-Ritz finite element model was developed, \( w = \tilde{N}^e_p \) and the following nonlinear equation system was given per elements:

\[ \tilde{K}(\eta(p, \xi), h(p))^e \cdot \tilde{P}^e = \tilde{f}_a(\eta(p, \xi), h(p))^e + \tilde{f}_b(\eta(p, \xi), h(p))^e - Q^e \]  

(36)

where:

\[ \tilde{K}^e = \int \Phi^e \cdot \tilde{P}^e \cdot \tilde{N}^e_p \cdot \tilde{N}^e_p \]  

(37)

\[ \tilde{f}_a^e = \int \tilde{f}_a \]  

(38)

\[ \tilde{f}_b^e = \int \tilde{f}_b \]  

(39)

\[ Q^e = \tilde{N}^e_p \cdot q^e \]  

(40)

\[ \tilde{N}^e_p = \frac{\partial \tilde{N}^e_p}{\partial x} \]  

(41)

The p-FEM model for the energy equation:

\[ \vartheta(\xi)^e = \frac{4}{1} M(\xi)^e_i \cdot T^e_i + \frac{4}{1} \sum_{k=1}^{p^e_{\text{d}}} \sum_{j=1}^{p^e_{\text{d}}} M(\xi)^e_{ij} \cdot \tilde{a}^e_{ij} + \sum_{k=1}^{b^e} M(\xi)^e_{j0} \cdot \tilde{P}^e_i \]  

(42)

\[ z(\xi)^e_0 = \frac{4}{1} \sum_{k=1}^{p^e_{\text{d}}} \sum_{j=1}^{p^e_{\text{d}}} M(\xi)^e_{ij} \cdot \tilde{a}^e_{ij} + \sum_{k=1}^{b^e} M(\xi)^e_{j0} \cdot \tilde{P}^e_i \]  

(43)

\[ x(\xi)^p_0 = \frac{4}{1} \sum_{k=1}^{p^e_{\text{d}}} \sum_{j=1}^{p^e_{\text{d}}} M(\xi)^e_{ij} \cdot \tilde{a}^e_{ij} + \sum_{k=1}^{b^e} M(\xi)^e_{j0} \cdot \tilde{P}^e_i \]  

(44)

where \( b^e \) is the inner approximation functions number, \( M_{ij} \) are the two dimensional Legendre functions according to Páczelt (1994) [16].

The equation system:

\[ \vec{\Theta}(u(p), v(p))^e \cdot \vec{T}^e = \vec{f}(u(p), v(p))^e + \vec{\Pi}^e \]  

(45)

where:

\[ \vec{\Theta}^e = \int \tilde{\Phi}^e \cdot \vec{u}^e \]  

(46)

\[ \vec{f}^e = \int \tilde{f}^e \cdot \vec{v}^e \]  

(47)

\[ \vec{\Pi}^e = \int \tilde{P}^e \cdot q^e \cdot ds^e \]  

(48)

9 CALCULATION

Newton-Raphson algorithm has been applied for solving the non-linear equation system. For illustration the Figure 1 and Figure 2 show the pressure and temperature distributions and the film shapes of a pure rolling and a rolling-sliding line contact. The line load is 100000 N/m, the reduced radius of the contact surfaces is R=18.18 mm and the lubricant is P-150 paraffin oil.
The order of the approximation can be found in the Table 1 and Table 2.

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<th>Element</th>
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<th>3</th>
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Table 1: Order of the pressure approximation

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<th>Element number in the upper row</th>
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<th>Element number in the middle row</th>
<th>Orders of the temperature approximation</th>
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Table 2: Orders for temperature approximation

<table>
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<tr>
<th>Figure 1.: Pressure distribution and film shape during pure rolling contact, $u_{surf}$ = 7 m/s.</th>
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<tbody>
<tr>
<td><img src="image1.png" alt="Figure 1" /> p [1e2MPa]</td>
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<td>h [µm]</td>
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<table>
<thead>
<tr>
<th>Figure 2.: Pressure distribution and film shape during rolling-sliding contact, $u_{sl}$ = 9 m/s, $u_{surf}$ = 5 m/s.</th>
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<td>h [µm]</td>
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10 REFERENCES


