HELLINGER-REISSNER MODEL AND ITS APPLICATION TO ELASTOMERIC MATERIALS IN CONTACT SITUATIONS

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SUMMARY
This paper presents the numerical modelling of a static restriction with an elastomeric O-Ring seal using the finite element method. To obtain the solution of this complex problem we need to solve two different problems: the elastic problem of large deformation theory and the contact problem with friction between the piston and the seal. The elastic problem is solved using a linearized mixed formulation (Hellinger-Reissner principle) while the contact zone is obtained from the Kombi’s algorithm. With this specific method we can define precisely the sliding and the adhesion zones and the friction. The basic equations and the assumptions used are presented. The finite element discretisation is developed here to obtain a numerical solution. We present in this paper an analytical validation of our model as well as a comparison with experimental results published elsewhere.

Keywords: Hellinger-Reissner, Elastomeric Material, Finite Element Method, Contact Algorithm.

1 INTRODUCTION
Friction forces and leakage levels must be known for the practical use of elastomeric-O-ring seals. Thus, a meaningful modelling of the contact between the O-ring seals, the piston and the sleeve is highly desirable. Figure 1 presents the goal of this modelling which is the determination of the O-ring behaviour subject to an axial gradient of pressure in static conditions.

Analysing this figure we can define two problems:

- an elasticity problem from a global point of view to obtain the strain and stress fields,
- a contact problem to determine the contact zone between the O-ring seal, the piston and the sleeve.

Very few works have been published concerning this complex problem. Nevertheless, we can cite the following papers which analyse this problem with analytical as well as numerical approaches using several sets of assumptions.

Johannesson and Kassfeldt [1] propose an analytical solution to a hydraulic case with the following assumptions:

- plane stress,
- non-linear behaviour which can take into account the decrease of compressibility with the pressure
- no friction at the walls.

This method based on the calculation of the stress field at the boundaries allows the determination of the pressure distribution and the contact zone.

Nevertheless, the calculated pressure distribution is found always higher than that obtained from experiments by Johannesson et al. [1] (figure 2). We can also cite the works of Dragoni and Strozzi in the same area [2], George et al. [3], Dragoni [4] using a finite element method to solve this problem in order to obtain a better accuracy of the results. However, all these works use certain restrictive assumptions. We present in this paper a numerical method based on the finite element and Hellinger-Reissner formulation to accurately solve the elasticity problem and the contact problem.
2 CONSTITUTIVE BEHAVIOUR OF THE MATERIAL

Although the seal has a non-linear behaviour from the elasticity point of view, we use here a linear law (Hooke's law), following the works of Treloar [5] and Johannesson [6] who show that this theory is valid if the strain field remains lower than 20 % (figure 3). From our experiments, we found a Young's modulus of 3.5 MPa and we have taken a Poisson coefficient of 0.49 (Dragoni [4]).

3 ELASTICITY FORMULATION

In the following development we consider the motion of a general body in a stationary Cartesian coordinate system as shown in figure 4, and assume that the body can experience large displacements, large strains and non-linear constitutive response.

We notice that the 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor are energetically conjugate. The following weak forms based on the Hellinger-Reissner principle [9], then are obtained:

\[(\text{Pichon [10]}) : \]

\[
\pi \left( \int^{t+\Delta t} o \pd{u}{t} + \int^{t+\Delta t} o \pd{S}{y} \right) = \frac{1}{2} \int_{\omega}^{o} \int^{t+\Delta t} o \pd{S}{y} \int^{t+\Delta t} o \pd{e}{y} \pd{d}{V}
\]

\[
- \int_{\omega}^{o} \int^{t+\Delta t} o \pd{u}{t} \int^{t+\Delta t} \pd{S}{y} \pd{d}{V} = \int_{\omega}^{o} \int^{t+\Delta t} o \pd{S}{y} \int^{t+\Delta t} o \pd{e}{y} \pd{d}{V}
\]

\[
- \frac{1}{2} \int_{\omega}^{o} \int^{t+\Delta t} o \pd{S}{y} \int^{t+\Delta t} o \pd{D}{-1} \int^{t+\Delta t} o \pd{S}{y} \pd{d}{V}
\]

\[
- \int_{\omega}^{o} \int^{t+\Delta t} o \pd{u}{t} \int^{t+\Delta t} \pd{T}{i} \pd{d}{S}
\]

The mixed weak form comes from the Hellinger-Reissner principles taking into account the large deformation we finally obtain [1]:

\[
\delta(\Delta u, \Delta S) = \int_{\omega}^{o} \int \delta(e + \Delta e) \left( (S + \Delta S) \right) \pd{d}{V} = \int_{\omega}^{o} \int \delta(u + \Delta u) \left( \pd{T}{+} + \Delta \pd{T}{+} \right) \pd{d}{S}
\]

\[
\int_{\omega}^{o} \int \frac{\delta(S + \Delta S)}{\delta e} \left( (S + \Delta S) \right) \pd{d}{V} - \int_{\omega}^{o} \int \frac{\delta(S + \Delta S)}{\delta e} \left( (S + \Delta S) \right) \pd{d}{V} = 0
\]

or \( \delta e = \delta S = \delta u = 0 \)

\[
\int_{\omega}^{o} \int \delta(\Delta e)(S + \Delta S) \pd{d}{V} = \int_{\omega}^{o} \int \delta(\Delta u) \left( \pd{T}{+} + \Delta \pd{T}{+} \right) \pd{d}{S}
\]

\[
- \int_{\omega}^{o} \int \frac{\delta(S + \Delta S)}{\delta e} \left( (S + \Delta S) \right) \pd{d}{V} = 0
\]

\[
\int_{\omega}^{o} \int \delta(\Delta e)(S + \Delta S) \pd{d}{V} + \int_{\omega}^{o} \int \delta(\Delta e) \left( (S + \Delta S) \right) \pd{d}{V} = \int_{\omega}^{o} \int \delta(\Delta u)(\Delta \pd{T}{+}) \pd{d}{S}
\]

\[
\int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} = 0
\]

\[
\int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} = \int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} = 0
\]

with \( \int_{\omega}^{o} \int \delta i^{i+1} = u^{i+1} + \Delta u \)

\[
\int_{\omega}^{o} \int S^{i+1} = S^{i+j} + \Delta S
\]

where \( i \) is the index for the incremental load.

To ensure the convergence of the numerical method used in the case of large deformations, we need linearisations neglecting terms of second order. We obtain:

\[
\delta(\Delta e) = \delta(\Delta e + \Delta \eta) = \delta(\Delta e)
\]

\[
\delta(\Delta u, \Delta S) =
\]

\[
\int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} + \int_{\omega}^{o} \int \delta(\Delta e) \left( (S + \Delta S) \right) \pd{d}{V} = \int_{\omega}^{o} \int \delta(\Delta e)(\Delta \pd{T}{+}) \pd{d}{S}
\]

\[
\int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} - \int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} = 0
\]

\[
\int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} = \int_{\omega}^{o} \int \delta(\Delta e)(\Delta S) \pd{d}{V} = 0
\]

The following non-linear algebraic systems, after discretisation by the finite element method are expressed in the weak form. Their solutions yields to displacement and stress fields.
4 THE CONTACT PROBLEM

It is possible to determine the contact zones between the seal and the other surfaces if the elasticity problem has been solved giving the displacement and the stress fields. Globally, the contact problem can be represented with a set of equalities and inequalities representing three physical conditions, (figure 5):

- a non penetration condition : \( d_n = 1 \) in the contact zone and \( d_n < 1 \) outside the contact zone with an allowed displacement value
- a contact condition \( T_n < 0 \) in the contact zone
- a friction condition using the Coulomb law :
  \[
  g = \mu |T_n|
  \]

If \( dt = 0 \iff |T_t| \leq g \) then we have adhesion, otherwise we have the two following conditions :

| \(|T_t| > g | \) : sliding

\[
 g \cdot dt < 0 : \text{the friction force is opposite the tangential displacement } \Rightarrow dt = 0.
\]

Either of two methods can be used to solve this complex problem : The first one is proposed by Paniagotopoulos and the second one (Kombi's method) by Kalker [11]. The second method is more interesting because it allows the simultaneous resolution of the normal and the tangential problems.

5 KOMBI'S METHOD

The following scheme resume this method (figure 6)

<table>
<thead>
<tr>
<th>Bit range</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 - If the node is situated on the left border</td>
</tr>
<tr>
<td>2</td>
<td>1 - If the node is situated on the right of the right border</td>
</tr>
<tr>
<td>3</td>
<td>1 - If the node is situated underneath the low zone</td>
</tr>
<tr>
<td>4</td>
<td>1 - If the node is situated underneath of the clearance</td>
</tr>
<tr>
<td>5</td>
<td>1 - If the node is situated above the high zone</td>
</tr>
</tbody>
</table>

Figure 6: Sketch of Kombi's numerical device

6 RESULTS

We compare results of our model with those obtained by George et al [3] considering the problem illustrated in figure 7.

Figure 8 shows a good correlation between these two methods. The following table shows that the linearisation has no influence for the determination of the contact zone.

In figure 9, the influence of Young’s modulus is presented. We can see that it is possible to reduce the differences of the pressure values in the contact zone if we increase Young’s modulus. These results show that our approach gives satisfactory results compared to a more complicated numerical codes. With these results, we can ensure that :

- the linearized system induces a very low error for the determination of the contact zone,
- assuming a linear law behavior can be useful if we increase Young’s modulus.
7 CONCLUSION

In this study, we have developed a new method to solve the numerical modeling of a static restriction with an elastomeric O-Ring seal using the finite element method. The elastic problem with the large deformation assumption as well as the contact problem have been presented. The elastic problem has been solved using the Hellinger-Reisner Principle while the contact zone has been obtained from Kombi's method. We showed first that two linearizations are necessary to avoid numerical instabilities, and second that we can optimize the contact zone with Young's Modulus. The results obtained are in good agreement compared to those in the literature and our approach seems to be easier to be implemented.

8 ACKNOWLEDGMENT

The authors would like to thank Fabrice Ville and Sebastien Baud, researchers in the Laboratoire de Mécanique des Contacts, for their technical contribution to this paper.

9 REFERENCES


10 NOMENCLATURE

D : matrix relative to the stress tensor and of the displacement field
ε : Green Lagrange strain tensor
Sij : 2nd Piola-Kirchoff stress tensor
T : external forces at the boundary
u : displacement field
π : functional
V : volume of the body
S : surface boundary
δ : variational operator
Δu : incremental value of the displacement
Δs : incremental value of the stress field
Kij : stiffness matrices