CONCENTRATED FORCE ACTING ON THE BOUNDARY OF AN ELASTIC HALF-PLANE WITH CIRCULAR HOLE

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SUMMARY
This paper deals with analytical determination of stresses which occur in an elastic half-plane that contains a hole and is loaded by normal and tangential point forces on its straight boundary. The expansions of elastic potentials expressed in bipolar co-ordinates in Fourier series, deduced by Barjansky and Evan-Iwanowski, are corrected and then the shear stresses are derived. The analysis of the stress state reveals the zone in which important stress riser effects occur and the values of this riser dependent on hole depth. It is found that if the hole is placed under a specified depth, its effect can be assessed by considering it to be placed in an elastic plane.

Keywords: elastic half-plane, concentrated force, stress state, elastic potentials, circular hole

1 INTRODUCTION
Fatigue life of rolling contacts encountered in the most important machine details depends on both subsurface and surface defects. The subsurface defects may occur as inner voids, inclusions, oxides, microcracks etc. These produce important stress risers and might become nuclei of fatigue initiation. The evaluation of stress riser effects, of great importance for life prediction, is a very difficult task for the actual state of contact mechanics. Therefore it is convenient to start the research with simpler situations such as plane problems. In this field, the first contribution was made by Jeffery, [1], back in 1920 who put the basis of plane elasticity in bipolar co-ordinates. In 1943 Barjansky, [2], applied Jeffery’s method to evaluate the stresses produced in a half-plane containing a hole by a normal or a tangential load concentrated in a point of straight boundary. As pointed out by Evan-Iwanowski, [3], in 1961, the results of Barjansky are not right because errors occurred when potentials were expanded into Fourier series. He proposed a correction of Barjansky results which still suffer of other errors as it will be shown herein. Tamate, [4], used the technique of complex potentials advanced by Muskhelishvili, to evaluate the stresses produced in a half-plane containing a circular hole by a flat-ended rigid punch that indents the straight boundary. In 1983, Miller and Keer, [5], presented a numerical solution to the two-dimensional problem of a rigid indenter sliding with friction on a half-plane containing a near surface imperfection in the form of a circular void or rigid inclusion. Being of a numerical nature, this solution does not allow generalisation. In order to offer an analytical general solution to the problem of a plane contact in the presence of a circular hole, a first step is performed in this paper by solving the Boussinesq and Cerruti plane problems by providing the required corrections to the works of Barjanski and Evan-Iwanowski.

2 FOURIER SERIES FOR ELASTIC POTENTIALS
A half-plane plane containing a circular hole as illustrated in Figure 1 is considered.

![Figure 1: The half-plane containing a circular hole](image)

The half plane is loaded by a normal \( N \) and a tangential \( T \) point forces. To find the stresses in a current point of the half-plane the Airy stress function \( \Phi(\xi, \eta) \) is used. This is expressed in bipolar co-ordinates \( \xi, \eta \) defined by following relations:

\[
x = -\frac{a \sin \eta}{ch \xi - \cos \eta}; \quad y = \frac{a \sin \eta}{ch \xi - \cos \eta}; \quad J = \frac{a}{ch \xi - \cos \eta}.
\]

As Savin, [6], shows it is convenient to express the stresses in bipolar co-ordinates by means of ratio \( \Phi(\xi, \eta) / J \), as follows:

\[
a \sigma_\xi = \left( (ch \xi - \cos \eta) \frac{\partial^2 \Phi}{\partial \eta^2} - sh \xi \frac{\partial \Phi}{\partial \xi} - \sin \eta \frac{\partial \Phi}{\partial \eta} + ch \xi \right) \Phi / J \;
a \sigma_\eta = \left( (ch \xi - \cos \eta) \frac{\partial^2 \Phi}{\partial \eta^2} - sh \xi \frac{\partial \Phi}{\partial \xi} - \sin \eta \frac{\partial \Phi}{\partial \eta} + \cos \eta \right) \Phi / J \;
a \tau_{\xi \eta} = -(ch \xi - \cos \eta) \frac{\partial^2 \Phi}{\partial \xi \partial \eta} \Phi / J.
\]

If the hole does not exist, the potentials \( \Phi_1 \) and \( \Phi_2 \) of the forces \( N \) and \( T \), respectively, in rectangular co-ordinates are known, [1]:
The problem was against analysed as follows. The indications clearly that this is not the case, which means simply a translation of the true potential. Figure 2 would be true, Eva-Iwanowski expansion would be the form of auxiliary potentials \( \chi_1 \) and \( \chi_2 \) are obtained by correcting the coefficients of Fourier series proposed by Barjansky, [2], and Evan-Iwanowsky, [3], because, as in Figure 2 shows, neither of these is right.

Rewritten in bipolar co-ordinates these become:

\[
\begin{align*}
\phi_1 &= \frac{N}{\pi} (y - y_0) \tan \left( \frac{y - y_0}{x} \right); \\
\phi_2 &= \frac{T}{\pi} x \tan \left( \frac{y - y_0}{x} \right).
\end{align*}
\]

(3)

where

\[
\begin{align*}
p &= \sin(\beta) \sh(\xi); & q &= \cos(\beta) \sh(\xi); \\
\psi &= \eta + \beta; & \beta &= \atan \left( \frac{y_0}{a} \right).
\end{align*}
\]

(5)

When the hole is present, \( \phi_1 \) and \( \phi_2 \) are replaced by \( \Phi_1 = \phi_1 + \chi_1 \) and \( \Phi_2 = \phi_2 + \chi_2 \) where \( \chi_1 \) and \( \chi_2 \) are auxiliary potentials satisfying the following conditions:

- to generate stresses that vanish in distant points \( (\xi \to 0, \eta \to 0) \);
- to generate stresses that vanish in the points of the straight line that bounds the half-plane, except the point of action of applied load;
- in combination with \( \phi_1 \) and \( \phi_2 \) to assure that in the points of hole contour both hoop and shear stresses vanish.

In this paper, the potentials \( \phi_1 \) and \( \phi_2 \) are obtained by correcting the coefficients of Fourier series proposed by Barjansky, [2], and Evan-Iwanowsky, [3], because, as in Figure 2 shows, neither of these is right.

\[
\begin{align*}
\phi_1 &= \frac{R_0}{2} + \sum_{k=1}^{\infty} \left[ R_k \cos(k\eta) + S_k \sin(k\eta) \right]; \\
\phi_2 &= \frac{T_0}{2} + \sum_{k=1}^{\infty} \left[ T_k \cos(k\eta) + U_k \sin(k\eta) \right],
\end{align*}
\]

(6)

where, for \( k = 0 \):

\[
R_0 = \frac{T}{2\pi} \sh \left[ 2\atan \left( \frac{p}{q} \right) + \pi - 2\beta - \ln \left( \frac{\sh(\beta - \xi)}{\sh(\beta + \xi)} \right) \right],
\]

whereas for \( k \geq 1 \) they become:

\[
R_k = \Re \left[ \frac{T}{\pi} \sh(\xi) e^{(\xi-i\beta)} \frac{i}{k} e^{i\beta} \left( -1 \right)^k e^{i\beta} \right];
\]

(7)

and

\[
T_0 = \frac{T}{2\pi} \sh \left[ 2\atan \left( \frac{p}{q} \right) + \pi - 2\beta - \ln \left( \frac{\sh(\beta - \xi)}{\sh(\beta + \xi)} \right) \right],
\]

where the coefficients for \( k > 1 \) are:

\[
T_k = - \frac{R_{k-1} \sin(\beta) - S_{k-1} \cos(\beta)}{2q} + \frac{p}{q} R_k - \frac{R_{k+1} \sin(\beta) + S_{k+1} \cos(\beta)}{2q};
\]

(8)

\[
U_k = - \frac{R_{k-1} \cos(\beta) + S_{k-1} \sin(\beta)}{2q} + \frac{p}{q} S_k - \frac{R_{k+1} \cos(\beta) - S_{k+1} \sin(\beta)}{2q}.
\]

The expressions for \( R_k \) and \( S_k \) entering expressions (8) are obtained by replacing \( T \) by \( N \) in equations (7).

The form of auxiliary potentials \( \chi_1 \) and \( \chi_2 \) are taken from Jeffery, [1], to be:

\[
\frac{X}{J} = B_0 \sh(\xi) - \left[ B_0 \left[ - \sh(\xi) \ch(\xi) \right] + 2H \sh(2\xi) + F \right] \cos(\eta) + \left[ G^2 \ch(2\xi) + H^2 \sin(\eta) \right] + \sum_{k=1}^{\infty} \left[ E_k \ch[(k+1)\xi] + F_k \sh[(k+1)\xi] + G_k \ch[(k-1)\xi] + H_k \sh[(k-1)\xi] \right] \cos(k\eta)
\]

(9)

In order to simplify the expressions of boundary conditions the potential \( \phi_1 / J \) and \( \phi_2 / J \) are written as:
\[
\Phi_1 = -\frac{P}{r} \cos \beta \tan(\beta) \cosh(\xi) + \sum_{k=1}^{\infty} \left[ \frac{p_k e^{-(k+1)\xi} + p_k' e^{(k+1)\xi}}{q_k e^{-(k+1)\xi} + q_k' e^{(k+1)\xi}} \right] \sin(\kappa) + \sum_{k=1}^{\infty} \left[ \frac{p_k' e^{-(k+1)\xi} - e^{-(k+1)\xi}}{q_k e^{-(k+1)\xi} - e^{-(k+1)\xi}} \right] \cos(\kappa) + \Phi_0 = r_0 \sinh(\xi) + \sum_{k=1}^{\infty} \left[ \frac{s_k e^{-(k+1)\xi} - e^{-(k+1)\xi}}{s_k e^{-(k+1)\xi} - e^{-(k+1)\xi}} \right] \sin(\kappa)
\]

3 Final Expressions of Elastic Potentials

In order to completely solve the problem, the following procedure is performed. First, the potentials \( \Phi_1 / J = \Phi_1 / J + \chi_1 / J \) and \( \Phi_2 / J = \Phi_2 / J + \chi_2 / J \) are built and the unknown coefficients entering the potentials are determined by taking into account the boundary conditions.

The stress function for the tangential load is:

\[
\Phi_2 = r_0 \sinh(\xi) + B_0 \cosh(\xi) + \frac{B_0 (\xi - \sinh(\xi) \cosh(\xi))}{2F_0 \sinh^2(\xi) - r_1 (1 - e^{2\xi})} \cos \eta + \frac{[G_0 \cosh(2\xi) + F_0' + q_1 (1 - e^{2\xi})] \sin(\eta) + \sum_{k=2}^{2} \left[ \begin{array}{c}
E_k (k-1) \sinh(\xi) \sinh(k\xi) + F_k' [k \sinh(\xi) \cosh(k\xi) - \cosh(\xi) \sinh(k\xi)] + \sin(\kappa) \\
E_k' (k-1) \sinh(\xi) \sinh(k\xi) + F_k' [k \sinh(\xi) \cosh(k\xi) - \cosh(\xi) \sinh(k\xi)] + \cos(\kappa) \\
S_k [e^{(k-1)\xi} - e^{-(k+1)\xi}] + S_k' [e^{(k-1)\xi} - e^{-(k+1)\xi}] \\
S_k [e^{(k-1)\xi} - e^{-(k+1)\xi}] + S_k' [e^{(k-1)\xi} - e^{-(k+1)\xi}]
\end{array} \right]}{2}
\]

where the coefficients for \( k < 2 \) are:

\[
B_0 = r_1 \frac{\sinh(\xi_0) - e^{\xi_0}}{\sinh(\xi_0)}; \quad F_0 = -r_1 \frac{e^{\xi_0}}{\sinh(\xi_0)}; \quad F_0' = 2 \frac{e^{2\xi_0} - 1}{\sinh(2\xi_0)}; \quad G_0' = S_1 \frac{e^{2\xi_0}}{\sinh(2\xi_0)}
\]

and for \( k \geq 2 \) they become:

\[
E_k = \frac{k \cosh(\xi_0) \sinh(\xi_0) - \sinh(\xi_0) e^{2\xi_0} + k^2 \sinh^2(\xi_0)}{k^2 \sinh^2(\xi_0) - \sinh(k^2 \xi_0)^2}; \quad F_k = k^2 \sinh^2(\xi_0) - \sinh(k^2 \xi_0)^2; \quad F_k' = k (k-1) \frac{\sinh(\xi_0)^2}{k^2 \sinh^2(\xi_0) - \sinh(k^2 \xi_0)^2} r_k
\]

For normal load, the stress function takes the form:

\[
\Phi_2 = B_0 \cosh(\xi) - \frac{P}{r} \cos \beta \tan(\beta) \cosh(\xi) - \frac{B_0 (\xi - \sinh(\xi) \cosh(\xi))}{2F_0 \sinh^2(\xi) + p_1 + P' e^{2\xi}} \cos \eta + \frac{[G_0' \cosh(2\xi) + F_0' + q_1 (1 - e^{2\xi})] \sin(\eta) + \sum_{k=2}^{2} \left[ \begin{array}{c}
E_k (k-1) \sinh(\xi) \sinh(k\xi) + F_k' [k \sinh(\xi) \cosh(k\xi) - \cosh(\xi) \sinh(k\xi)] + \sin(\kappa) \\
E_k' (k-1) \sinh(\xi) \sinh(k\xi) + F_k' [k \sinh(\xi) \cosh(k\xi) - \cosh(\xi) \sinh(k\xi)] + \cos(\kappa) \\
S_k [e^{(k-1)\xi} - e^{-(k+1)\xi}] + S_k' [e^{(k-1)\xi} - e^{-(k+1)\xi}] \\
S_k [e^{(k-1)\xi} - e^{-(k+1)\xi}] + S_k' [e^{(k-1)\xi} - e^{-(k+1)\xi}]
\end{array} \right]}{2}
\]

where the coefficients for \( k < 2 \) are:

\[
B_0 = -\frac{P \cos^2(\beta) \cosh^2(\xi)}{\sinh(\xi_0)}; \quad F_1 = -\frac{P \cos^2(\beta)}{\sinh(\xi_0)}; \quad G_1' = 0; \quad F_1' = q_1 \quad (15.1)
\]

and for \( k \geq 2 \) they become:

\[
E_k = \frac{1}{2} \left[ \begin{array}{c}
p_k (k-1) \sinh(\xi_0) e^{-\xi_0} - \sinh(k\xi_0) e^{2\xi_0} + p_k' (k-1) \sinh(\xi_0) e^{2\xi_0} - \sinh(k\xi_0) e^{-\xi_0} + \sin(\kappa) \\
p_k (k-1) \sinh(\xi_0) e^{-\xi_0} - \sinh(k\xi_0) e^{2\xi_0} + p_k' (k-1) \sinh(\xi_0) e^{2\xi_0} - \sinh(k\xi_0) e^{-\xi_0} + \cos(\kappa)
\end{array} \right]
\]

\[
F_k = \frac{k-1}{2} \left[ \begin{array}{c}
p_k \sinh(k\xi_0) e^{2\xi_0} + k \sinh(\xi_0) e^{-\xi_0} + q_k \sinh(\xi_0) e^{-\xi_0} - k \sinh(k\xi_0) e^{2\xi_0} + \sin(\kappa) \\
p_k \sinh(k\xi_0) e^{2\xi_0} + k \sinh(\xi_0) e^{-\xi_0} + q_k \sinh(\xi_0) e^{-\xi_0} - k \sinh(k\xi_0) e^{2\xi_0} + \cos(\kappa)
\end{array} \right]
\]

\[
E_k' = \frac{1}{2} \left[ \begin{array}{c}
q_k [k \sinh(\xi_0) e^{-\xi_0} - \sinh(k\xi_0) e^{2\xi_0}] + q_k' [k \sinh(\xi_0) e^{-\xi_0} - \sinh(k\xi_0) e^{2\xi_0}] + \sin(\kappa) \\
q_k [k \sinh(\xi_0) e^{-\xi_0} - \sinh(k\xi_0) e^{2\xi_0}] + q_k' [k \sinh(\xi_0) e^{-\xi_0} - \sinh(k\xi_0) e^{2\xi_0}] + \cos(\kappa)
\end{array} \right]
\]

\[
F_k' = \frac{k-1}{2} \left[ \begin{array}{c}
q_k [k \sinh(k\xi_0) e^{2\xi_0} + k \sinh(\xi_0) e^{-\xi_0}] + q_k' [k \sinh(k\xi_0) e^{2\xi_0} + k \sinh(\xi_0) e^{-\xi_0}] + \sin(\kappa) \\
q_k [k \sinh(k\xi_0) e^{2\xi_0} + k \sinh(\xi_0) e^{-\xi_0}] + q_k' [k \sinh(k\xi_0) e^{2\xi_0} + k \sinh(\xi_0) e^{-\xi_0}] + \cos(\kappa)
\end{array} \right]
\]
\[ \sigma_x = \frac{\left[ \text{ch}(\xi) \cos(\eta) - 1 \right]^2 \sigma_{\eta\eta} + \text{sh}(\xi) \sin(\eta) \sigma_{\eta\epsilon} + 2\text{sh}(\xi) \sin(\eta) \left[ \text{ch}(\xi) \cos(\eta) - 1 \right] \tau_{\eta\epsilon} }{\left[ \text{ch}(\xi) - \cos(\eta) \right]^2}; \]

\[ \sigma_y = \frac{\left[ \text{sh}(\xi) \sin(\eta) \sigma_{\eta\eta} + \left[ \text{ch}(\xi) \cos(\eta) - 1 \right]^2 \sigma_{\eta\epsilon} + (-1)2\text{sh}(\xi) \sin(\eta) \left[ \text{ch}(\xi) \cos(\eta) - 1 \right] \tau_{\eta\epsilon} }{\left[ \text{ch}(\xi) - \cos(\eta) \right]^2}; \]

\[ \tau_{\eta\epsilon} = \frac{\left[ \text{sh}(\xi) \sin(\eta) \left[ \text{ch}(\xi) \cos(\eta) - 1 \right] \sigma_{\eta\eta} - \left( \text{sh}(\xi) \sin(\eta) \right)^2 \tau_{\eta\epsilon} \right]}{\left[ \text{ch}(\xi) - \cos(\eta) \right]^2}. \]

In a general case the following potential is formed:

\[ \Phi = \frac{\alpha \Phi_1 + \beta \Phi_2}{J}, \quad (17) \]

which can take the following particular forms:

- \( \alpha = 1, \ \beta = 0 \) for normal load;
- \( \alpha = 0, \ \beta = 1 \) for tangential load;
- \( \alpha = 1, \ \beta = \mu \) for contact with Coulomb friction.

The stresses can be calculated by equations (2).

A special care must be accorded to the number of terms in potentials expansions due to low convergence in the neighbourhood of the straight boundary. A special program has been developed to assess the minimum number of terms that assures a desired precision.

Figure 3 shows the maximum shear stress into the half-plane when a point force acts normally and passes through the hole centre. Two situations were chosen, first in the absence of friction and the second for a quite high friction coefficient, \( \mu = 0.8 \).

The variation of hoop stress with the depth at which the hole is position is illustrated in Figure 4. The traces shown indicate that the riser effect of the hole occurs in the point sited at a minimum distance from the straight boundary of the half-plane. This maximum value increases as the hole gets closer to this boundary.

In order to evidence the dependence of hoop stress on the hole position depth \( h_0 \), Figure 5 was traced.

4 CONCLUSIONS

This paper solves analytically the two fundamental problems, Boussinessq and Cerruti, of the half-plane with a circular hole. These results allow to treat the effects of a load distributed along the straight boundary by using the principle of superposition.

In the neighbourhood of load point the stresses are important but of a lower intensity than in the case of a compact half-space because the hole reduces the local stiffness in this zone.

The hole induces a high stress riser effect. This is very high when the depth of hole positioning is small an decreases rapidly with the increase of this depth. Practically, at a higher depth then five hole radii, the stresses are the same as in the situation when the hole is placed in an elastic plane loaded at infinity.

5 REFERENCES