The problem of stress concentration around inner defects in contacting bodies is a major subject of contact mechanics. In order to assess the stress riser effects, these authors made the first step in a previous work by deriving analytical expressions for the stresses generated in a elastic half-plane possessing a hole and point loaded on its boundary. The next step is the subject of this paper. The previous results are used to derive, by applying the superposition principle, the stress state produced in the same half-plane by half-elliptical distributions of tractions. The analysis reveals important stress features depending on hole dimension, hole depth and the loading width.

Keywords: elastic half-plane, circular hole, stress state, friction, semi-elliptical traction

1 INTRODUCTION

In the a wider subject of stress concentration generated in contacting bodies by inner defects, recently, Alaci and Diaconescu [1], solved analytically the fundamental problem of an elastic half-plane containing a hole and loaded by point normal and tangential forces applied in the same point of the straight boundary. As a naturally continuation, the results obtained by them are used herein to evaluate the stress state generated in a similar plane by a half-elliptical oblique traction loaded on the boundary. The inclined traction resolves in normal traction or pressure and tangential traction or friction. In order to evaluate the stresses produced by this load, the principle of superposition is applied to the results reported in [1].

2 CALCULATION METHOD

The elastic half-plane containing a hole and the half-elliptical load is illustrated in Figure 1.

Figure 1: The half plane having a circular hole loaded by elliptical traction

The normal load, assumed half-elliptical, is applied over a width 2b, its distribution being centred at a horizontal distance e of hole centre. Therefore, by denoting with s the abscissa of a loaded point, the pressure equation can be written as:

\[ p(s) = p_0 \sqrt{1 - \left(\frac{s-e}{2b}\right)^2} \]  

Assuming a Coulomb friction the tangential traction distribution is described by following equation:

\[ t(s) = \mu p(s), \]

\[ \mu \] being the coefficient of friction.

According to the superposition principle, if \( \sigma_i(x, y) \) is any stress generated by normal or tangential point force, and expressed as:

\[ \sigma_i(x, y) = (P \text{ or } T) \cdot g_i(x, y), \]

then the corresponding stress induced by a traction distribution takes the following relation:

\[ \sigma_e(x, y) = \int_{e-b}^{e+b} (p(s) \text{ or } t(s)) g_i(x, y - s) \, ds \]  

Therefore, the evaluation of stress components reduces to calculation of integrals entering equation (4).

In order to find the main tensile or shear stress, respectively \( \sigma_{\text{max}}, \sigma_{\text{min}} \) and \( \tau_{\text{max}} \), the following procedure is adequate:

- the basic stresses corresponding to point forces are evaluated by using the method described in [1] and the corresponding functions \( g_i(x, y) \) are identified;

- by replacing in equation (4) the adequate traction and expression \( g_i(x, y - s) \) and performing the integration the analysed stresses are found; the stresses produced by normal and tangential load can be evaluated individually or simultaneously by using either each of two potentials or their sum;

- after building the stress tensor in bipolar or rectangular co-ordinates, the main tensile stresses are obtained as eigenvalues of this tensor; the main shear stresses are the half-sum of corresponding main tensile stresses.
The main stresses are given by following expressions:

\[ \sigma_{\text{max}} = \frac{\sigma_\alpha + \sigma_\beta}{2} + \sqrt{\left(\frac{\sigma_\alpha - \sigma_\beta}{2}\right)^2 + \tau_{\alpha\beta}^2}; \]

\[ \sigma_{\text{min}} = \frac{\sigma_\alpha + \sigma_\beta}{2} - \sqrt{\left(\frac{\sigma_\alpha - \sigma_\beta}{2}\right)^2 + \tau_{\alpha\beta}^2}; \]

\[ \tau_{\text{max}} = \sqrt{\left(\frac{\sigma_\alpha - \sigma_\beta}{2}\right)^2 + \tau_{\alpha\beta}^2}. \] (5)

In order to obtain the 3-D plots of stresses it is required to express them in rectangular co-ordinates. To this end the following inverse transform must be used:

\[ x = \frac{a \sin(\xi)}{\cosh(\xi) - \cos(\eta)}; \]

\[ y = \frac{a \sinh(\xi)}{\cosh(\xi) - \cos(\eta)}. \] (6)

The equations (6) are transcendental and therefore they are solved numerically.

When computing the integrals (4) by aid of MATHCAD the time required for evaluating one of the stresses is excessively long. Therefore, the integral was replaced by a sum as follows:

\[ \sigma_{ij}(x, y) \approx \frac{2 b}{n} \sum_{k=0}^{n-1} \Delta x \Delta y \sigma_{ij}(x, y, \frac{s_k + s_{k+1}}{2}). \] (7)

where \( n \) is the number of intervals in which the integration domain was divided into.

The results were saved as matrices. In order to plot the results 100 equidistant points were chosen on \( x \) and \( y \) directions. In order to speed up the calculation only 12 terms were considered in the series expansions. Even in these conditions the calculation time exceeds 14 hours for tangential loads and 18 hours for normal load when using a Pentium II/400MHz computer. Again, special measures were taken to assure the required precision and the stability of solution.

A typical set of input data for the program is illustrated in Figure 2.

![Figure 2: Typical set of input data](image)

**3 NUMERICAL RESULTS**

An exhaustive program of numerical analysis was carried out to evidence the effect of various factor, upon the stress state. Several of these results are given herein.

Figures 3 and 4 show the variation of maximum shear stress and maximum tensile stress, respectively, in the following situations: a) half elliptical normal load; b) half elliptical tangential load; c) half elliptical normal load in the presence of Coulomb friction for a friction coefficient \( \mu = 0.5 \). This value of \( \mu \) is rather high but was chosen to better evidence the effect of friction. The stresses are plotted as contour lines.
In order to check the validity of the above procedure the known example of plane Hertz problem is chosen. In Figure 5 the maximum shear stress occurring in this problem is plotted in three situations, namely, a) the exact solution, b) the solution obtained in bipolar co-ordinates for $n = 20$, $N = 12$ and $n_x = n_y = 100$, c) the solution obtained in bipolar co-ordinates for $n = 100$, $N = 32$ and $n_x = n_y = 100$.

It is obvious that, excepting some supplementary stress nuclei placed in the neighbourhood of boundary straight line, the Figures 5a and 5c are practically the same.
The stress riser effect generated by the hole deserves special attention. On the hole contour the only variable stress is the hoop stress \( \sigma_{\eta h} \). The variation of this stress along the hole circumference is seen in Figure 6.

Two families of hoop stress curves are shown comparatively in Figure 8.

In this figure, the curves a, b and c correspond, respectively, to an equivalent point load, to a half-elliptical load over a half-width \( b = 2r_0 \) and to a half-elliptical load over a half-width \( b = 10r_0 \). The curves d, e, and f are representative for the hoop stresses occurring in the points of a circle identical with the hole contour but placed in a compact half-plane.

Figure 7 shows the hoop stresses for the same hole but placed at a depth \( h_0 = 50r_0 \).

It can be noticed that each of the families of curves a, b, c and d, e, f reduce to a single curve. This indicates that as such depth, according to Saint Vénant principle, the actual distribution of the load is of no consequence.

4 CONCLUSIONS

The following conclusions can be drawn from the above work:

For central normal loading, the maximum shear stress possesses two in-depth symmetric maxima, placed symmetrically with respect to the load direction.

For holes placed at large depths, the hoop stress does not depend on the actual load application and is the same as in the situation the hole is placed un an elastic plane loaded at infinity.

As the hole radius diminishes with respect to the half width \( b \) the amplitude of hoop stress around the hole decreases.

If the hole is placed near the surface, a wide range of values for the ratio \( b / r_0 \) occurs for which the hoop stress on hole contour remains compressive. This situation is favourable for contact fatigue. On the other hand, as the depth hole position increases, the hoop stresses decrease rapidly. It can be concluded that there is an optimum value of defect position that depends on its dimension and the width of contact area.

5 REFERENCES